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THE LOCAL THETA CORRESPONDENCE FOR UNRAMIFIED UNITARY GROUPS

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ABSTRACT

We study the local theta correspondences for dual reductive pairs consisting of quasi-split unitary groups defined over a non-archimedean local field. We construct Howe's correspondence between the set of spherical representations of the one group and that of the other group by using the Whittaker model.

Introduction

Let $G_n^* = U(n, n+1)$ and $G_n = U(n, n)$ be quasi-split unitary groups defined over a global field k. In [6], we calculated some Fourier coefficients of an automorphic form $\varphi^* = {}^1\theta^n(\varphi|f)$ on $G_n^*(\mathbf{A})$ obtained from the global theta lifting of a cusp form φ on $G_n(\mathbf{A})$. In particular, we proved that a Whittaker function W_{φ^*} of φ^* is represented by a convolution of a Whittaker function W_φ of φ and a certain function $\Psi(f)$ defined from a Schwartz–Bruhat function f ([6, Corollary 5.5]). This formula is roughly written as

$$W_{\varphi^*}(h) = \int_{U_n(\mathbf{A})\backslash G_n(\mathbf{A})} W_\varphi(g)\Psi(\omega(h)f)(g)dg, \tag{0.1}$$

where ω is a Weil representation and U_n a maximal unipotent subgroup of G_n. Since the integral of the right-hand side is decomposed to an Euler product, we can consider the analogous formula for each local field k_v. The purpose of this paper is to calculate the unramified local factors of the integral of (0.1).

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To be more precise, let F be a non-archimedean local field and assume that both G_n^* and G_n are defined over the ring $\mathcal{O}$ of integers of F. Let η be an unramified quasi-character of a Borel subgroup of $G_n(F)$, and W_η a corresponding unramified Whittaker function. If a Schwartz–Bruhat function f_0 is invariant by the action of $G_n^*(\mathcal{O}) \times G_n(\mathcal{O})$, then the integral

$$(W_\eta, \Psi(\omega(h)f_0)) = \int_{U_n(F)\backslash G_n(F)} W_\eta(g)\Psi(\omega(h)f_0)(g)dg$$

gives an unramified Whittaker function on $G_n^*(F)$. Our result is a determination of the unramified quasi-character η^* of a Borel subgroup of $G_n^*(F)$ associated to this unramified Whittaker function (Proposition 2.2). If π_η and $\pi^*_{\eta^*}$ denote irreducible spherical representations of parameters η and η^*, respectively, then our result implies $\mathrm{Hom}_{G_n^*(F)\times G_n(F)}(\omega, \pi^*_{\eta^*} \otimes \pi_\eta) \neq 0$. In other words, the correspondence $\pi_\eta \mapsto \pi^*_{\eta^*}$ realizes the local Howe correspondence. It should be noted that Howe proved in [3] that spherical representations correspond to spherical representations in all unramified dual reductive pairs. However, he did not give any information about the matching of parameters.

A similar result is proved for the pair (G_n^*, G_{n+1}) in Section 3. The method used in this paper can also be applied to the dual reductive pair (GL_n, GL_{n+1}). We will study this Type 2 case in a forthcoming paper.

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Notation

For an associative ring R with identity element, we denote by $R^\times$ the group of all invertible elements of R and by $M_n(R)$ the set of all $n \times n$ matrices with entries in R. For $A \in M_n(R)$, ${}^tA$, $\mathrm{Tr}A$ and $\det A$ stand for its transpose, trace and determinant. If Z is an R-module and $x_1, x_2, \ldots, x_k$ are elements in Z, the submodule generated by $x_1, x_2, \ldots, x_k$ is denoted by $\langle x_1, x_2, \ldots, x_k\rangle$.

Let F be a p-adic field and E the unramified quadratic extension of F. We assume $p \neq 2$ because we use the results of Howe [3, Theorems 7.1, 10.2]. The norm and the trace of E over F is denoted by $\mathrm{N}_{E/F}$ and $\mathrm{tr}_{E/F}$, respectively. Let $\mathcal{O}$ (resp. $\mathcal{O}_E$) be the ring of integers of F (resp. E), ϖ a prime element of F and q the order of $\mathcal{O}/\varpi\mathcal{O}$. Then the order q_E of $\mathcal{O}_E/\varpi\mathcal{O}_E$ equals q^2. The absolute valuation of F (resp. E) is denoted by $|\cdot|_F$ (resp. $|\cdot|_E$), which is normalized as $|\varpi|_E = |\mathrm{N}_{E/F}(\varpi)|_F = q_E^{-1}$. For each $a \in E$, $\bar{a}$ stands for the image of a by the Galois involution of E over F. We fix a non-trivial additive character μ of F with the conductor $\mathcal{O}$. Then $\mu_E = \mu \circ \mathrm{tr}_{E/F}$ is a non-trivial additive character of E with the conductor $\mathcal{O}_E$.

For a connected linear algebraic group G defined over F, we denote by $G(F)$ the group of F-rational points. If G is further unramified, $G(\mathcal{O})$ stands for the group of $\mathcal{O}$-rational points. We normalize an invariant measure on $G(F)$ as the volume of $G(\mathcal{O})$ equals 1.

1. Unramified Whittaker functions of quasi-split unitary groups

First, we define some notations, which are slightly different from [6]. Let Z_n^* be a $2n+1$ dimensional vector space over E with a basis $\{e_1^*, \cdots, e_n^*, e_0^*, f_1^*, \cdots, f_n^*\}$ and $(,)_n$ the Hermitian form on Z_n^* represented by the matrix

$$J_n^* = \begin{pmatrix} 0 & 0 & 1_n \\ 0 & 1 & 0 \\ 1_n & 0 & 0 \end{pmatrix}.$$

Both subspaces $X_n^* = \langle e_1^*, \ldots, e_n^* \rangle$ and $Y_n^* = \langle f_1^*, \ldots, f_n^* \rangle$ are maximally isotropic. Let G_n^* denote the automorphism group of $(Z^*, (,)_n)$, that is

$$G_n^*(F) = \{g \in GL_{2n+1}(E) \colon {}^t g J_n^* \overline{g} = J_n^*\}.$$

We define algebraic subgroups of G_n^* as

T_n^* = maximal torus consisting of diagonal matrices in G_n^*,

P_n^* = stabilizer of the full isotropic flag $\langle e_1^* \rangle \subset \langle e_1^*, e_2^* \rangle \subset \cdots \subset X_n^*$,

U_n^* = unipotent radical of P_n^*.

On the other hand, Z_n denotes a $2n$ dimensional vector space over E with a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ equipped with skew Hermitian form $\langle , \rangle_n$ represented by the matrix

$$J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

We put $X_n = \langle e_1, \ldots, e_n \rangle$, $Y_n = \langle f_1, \ldots, f_n \rangle$ and

$$G_n(F) = \{g \in GL_{2n}(E) \colon {}^t g J_n \overline{g} = J_n\}.$$

Likewise as above, we define

T_n = maximal torus consisting of diagonal matrices in G_n,

P_n = stabilizer of the full isotropic flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset X_n$,

U_n = unipotent radical of P_n.

In the following, $G_n^{(*)}$ (resp. $T_n^{(*)}, P_n^{(*)}, \ldots$) stands for either one of the groups G_n^* or G_n (resp. T_n^* or T_n, P_n^* or P_n, ...). This convention is also used for other notations. Namely, if $\mathbf{X}^*$ is an object with respect to G_n^* and $\mathbf{X}$ a corresponding object for G_n, then $\mathbf{X}^{(*)}$ denotes either one of the objects $\mathbf{X}^*$ or $\mathbf{X}$.

We recall the explicit formulas of unramified Whittaker functions. For each $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbf{Z}^n$, we denote by $t_{\mathbf{k}}^{(*)}$ the diagonal matrix in $T_n^{(*)}(F)$ whose i-th diagonal entry is ϖ^{k_i} for $1 \le i \le n$. Further, for each $\mathbf{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n$, we define the unramified character $\eta_{\mathbf{z}}^{(*)}$ of $T_n^{(*)}(F)$ by

$$\eta_{\mathbf{z}}^{(*)}(t_{\mathbf{k}}^{(*)}) = z_1^{k_1} \cdots z_n^{k_n} .$$

This correspondence identifies $(\mathbb{C}^\times)^n$ with the set of unramified characters of $T_n^{(*)}(F)$, and hence the action of the Weyl group of $G_n^{(*)}$ on $T_n^{(*)}$ induces the action on $(\mathbb{C}^\times)^n$. We fix a closed Weyl chamber of the form

$$\Omega_n = \{\mathbf{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n \colon |z_1| \le |z_2| \le \cdots \le |z_n| \le 1\}.$$

Let $I_{\mathbf{z}}^{(*)} = \mathrm{Ind}_{P_n^{(*)}(F)}^{G_n^{(*)}(F)} \eta_{\mathbf{z}}^{(*)}$ be the normalized induced representation, that is, the set of all locally constant functions $\varphi\colon G_n^{(*)}(F) \to \mathbb{C}$ such that $\varphi(tug) = \eta_{\mathbf{z}}^{(*)}(t)\delta_n^{(*)}(t)^{1/2}\varphi(g)$ for all $t \in T_n^{(*)}(F)$, $u \in U_n^{(*)}(F)$ and $g \in G_n^{(*)}(F)$. Here, modular characters δ_n^* and δ_n are given as

$$\delta_n^{(*)}(t_{\mathbf{k}}^{(*)}) = \begin{cases} \prod_{i=1}^n |\varpi|_E^{2(n-i+1)k_i} & \text{if } G_n^{(*)} = G_n^*, \\ \prod_{i=1}^n |\varpi|_E^{(2n-2i+1)k_i} & \text{if } G_n^{(*)} = G_n . \end{cases}$$

Let $\varphi_{\mathbf{z}}^{(*)}$ be a non-zero $G_n^{(*)}(\mathcal{O})$ invariant function in $I_{\mathbf{z}}^{(*)}$ and $\psi^{(*)}$ be an unramified principal character of $U_n^{(*)}(F)$. We denote by $W_{\mathbf{z}}^{(*)}$ the image of $\varphi_{\mathbf{z}}^{(*)}$ by a unique (up to constant) non-zero $G_n^{(*)}(F)$-morphism from $I_{\mathbf{z}}^{(*)}$ to $\mathrm{Ind}_{U_n^{(*)}(F)}^{G_n^{(*)}(F)}\psi^{(*)}$. This $W_{\mathbf{z}}^{(*)}$ is holomorphic in $z \in (\mathbb{C}^\times)^n$ and determined by its restriction to $\{t_{\mathbf{k}}^{(*)} \colon \mathbf{k} \in \mathbb{Z}^n\}$. In order to describe $W_{\mathbf{z}}^{(*)}(t_{\mathbf{k}}^{(*)})$ explicitly, we use the following notation. For $z \in \mathbb{C}^\times$ and $k \in \mathbb{Z}$, we define $K(z, k/2)$ by

$$K(z, k/2) = z^{k/2} - z^{-k/2} .$$

Here the argument of $z^{1/2}$ is taken as $-\pi/2 < \arg z^{1/2} \le \pi/2$. Let

$$\Lambda_n = \{\mathbf{k} \in \mathbb{Z}^n \colon k_1 \ge k_2 \ge \cdots \ge k_n \ge 0\} ,$$

$$\kappa^{(*)} = \begin{cases} 1 & \text{if } G_n^{(*)} = G_n^*, \\ 1/2 & \text{if } G_n^{(*)} = G_n . \end{cases}$$

Furthermore, for $\mathbf{z} \in (\mathbb{C}^\times)^n$, let

$$\zeta^*(\mathbf{z}) = \prod_{i=1}^{n}(1 - q_E^{-1}z_i)(1 + q^{-1}z_i) \prod_{1\le i<j\le n} (1 - q_E^{-1}z_i z_j^{-1})(1 - q_E^{-1}z_i z_j) ,$$

$$\zeta(\mathbf{z}) = \prod_{i=1}^{n}(1 - q^{-1}z_i) \prod_{1\le i<j\le n} (1 - q_E^{-1}z_i z_j^{-1})(1 - q_E^{-1}z_i z_j) ,$$

$$\epsilon^{(*)}(\mathbf{z}) = \prod_{1\le i<j\le n} (z_i - z_j)(1 - z_i^{-1}z_j^{-1}) \prod_{i=1}^{n} K(z_i, \kappa^{(*)}) .$$

In this paper we normalize $W_{\mathbf{z}}^{(*)}$ as in [1], i.e. as $W_{\mathbf{z}}^{(*)}(1) = \zeta^{(*)}(\mathbf{z})$. Then, for each $z \in (\mathbb{C}^\times)^n$, a formula of Casselman and Shalika shows that if $\mathbf{k} \in \Lambda_n$, then

$$(1.1)\quad W_{\mathbf{z}}^{(*)}(t_{\mathbf{k}}^{(*)}) = \frac{\zeta^{(*)}(\mathbf{z})}{\epsilon^{(*)}(\mathbf{z})}\delta_n^{(*)}(t_{\mathbf{k}}^{(*)})^{1/2} \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} K(z_{\sigma(i)}, k_i+(n-i)+\kappa^{(*)}) ,$$

otherwise $W_{\mathbf{z}}^{(*)}(t_{\mathbf{k}}^{(*)})$ equals 0. Here S_n denotes the n-th symmetric group.

Let $\pi_{\mathbf{z}}^{(*)}$ be the unique irreducible spherical constituent of $I_{\mathbf{z}}^{(*)}$. We call $\pi_{\mathbf{z}}^{(*)}$ generic if it admits a Whittaker model. By [4, Theorem 2.2], it is known that $\pi_{\mathbf{z}}^{(*)}$ is generic if and only if $\zeta^{(*)}(\mathbf{z})\zeta^{(*)}(\mathbf{z}^{-1}) \neq 0$. Let $\mathbf{W}_{\mathbf{z}}^{(*)}(\psi^{(*)})$ denote the $G_n^{(*)}(F)$-module generated by $W_{\mathbf{z}}^{(*)}$. Obviously, if $\pi_{\mathbf{z}}^{(*)}$ is generic, it is isomorphic to $\mathbf{W}_{\mathbf{z}}^{(*)}(\psi^{(*)})$. In general, $\pi_{\mathbf{z}}^{(*)}$ is isomorphic to the unique irreducible quotient of $\mathbf{W}_{\mathbf{z}}^{(*)}(\psi^{(*)})$ if $\mathbf{z} \in \Omega_n$ (cf. [4, Section 2]).

Finally, we recall the Weil representations of the unitary group $G_m(F)$. Considering Z_m as a vector space over F equipped with symplectic form $\mathrm{tr}_{E/F}(\langle,\rangle_m)$, $G_m(F)$ is embedded in $Sp_{4m}(F)$. Let $Mp_{4m}(F) \to Sp_{4m}(F)$ be the metaplectic cover and ω_μ the Weil representation of $Mp_{4m}(F)$ associated with μ. If ν is a character of $E^\times$ whose restriction to $F^\times$ gives the non-trivial character of $F^\times/\mathrm{N}_{E/F}(E^\times)$, then there exists a splitting $s_{\mu,\nu}$: $G_m(F) \to Mp_{4m}(F)$ ([2, Proposition 3.1.1]). The representation $\omega_\mu \circ s_{\mu,\nu}$ of $G_m(F)$ is denoted by $\omega_{\mu,\nu}^m$, which acts on the space $\mathcal{S}(Y_m)$ of Schwartz–Bruhat functions on Y_m as

$$\omega_{\mu,\nu}^m\left(\begin{pmatrix} A & 0 \\ 0 & {}^t\overline{A}^{-1}\end{pmatrix}\right) f(\vec{x}) = \nu(\det A)|\det A|_E^{1/2} f({}^t\overline{A}\vec{x}), \quad (A \in GL_m(E)),$$

$$\omega_{\mu,\nu}^m\left(\begin{pmatrix} 1_m & B \\ 0 & 1_m\end{pmatrix}\right) f(\vec{x}) = \mu({}^t\overline{\vec{x}}B\vec{x})f(\vec{x}), \quad (B = {}^t\overline{B} \in M_m(E)).$$

In this paper, we fix ν as the non-trivial unramified quadratic character of $E^\times$, that is, $\nu(\varpi) = -1$ and $\nu|_{\mathcal{O}_E^\times} = 1$.

2. The local theta correspondence from G_n to G_n^*

If we consider the space $Z_n^* \otimes Z_n$ equipped with skew Hermitian form $(,)_n \otimes \langle,\rangle_n$, then $U(1)\backslash G_n^*(F) \times G_n(F)$ is embedded in $G_{n(2n+1)}(F)$, where $U(1)$ denotes the central torus $\{(t1_{2n+1}, \bar{t}1_{2n}) \colon t \in \ker \mathrm{N}_{E/F}\}$, and hence, the Weil representation $\omega_{\mu,\nu}^{n(2n+1)}$ is restricted to $G_n^*(F) \times G_n(F)$. Throughout this section, we write simply ω for $\omega_{\mu,\nu}^{n(2n+1)}$. We take a totally isotropic subspace $Y_{n(2n+1)}$ of $Z_n^* \otimes Z_n$ as

$$Y_{n(2n+1)} = Y_n^* \otimes Z_n + \langle e_0^* \rangle \otimes Y_n = \bigoplus_{i=1}^{n} f_i^* \otimes Z_n + e_0^* \otimes Y_n \ ,$$

which is naturally identified with $(Z_n)^n \oplus Y_n$. The action of $G_n^*(F) \times G_n(F)$ on $\mathcal{S}((Z_n)^n \oplus Y_n)$ is given as follows. For $(\vec{x}; y) = (x_1, \ldots, x_n; y) \in (Z_n)^n \oplus Y_n$ and a column vector $\alpha \in E^n$, we set

$$B_\alpha = \begin{pmatrix} -\alpha {}^t\overline{\alpha}/2 & -\alpha \\ -{}^t\overline{\alpha} & 0 \end{pmatrix} \in M_{n+1}(E) \ ,$$

$$\mathrm{Gr}_n(\vec{x}) = \begin{pmatrix} \langle x_1, x_1 \rangle_n & \cdots & \langle x_1, x_n \rangle_n \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle_n & \cdots & \langle x_n, x_n \rangle_n \end{pmatrix} \in M_n(E) \ ,$$

$$\mathrm{Gr}_{n+1}^*(\vec{x}; y) = \begin{pmatrix} (x_1, x_1)_{n,0} & \cdots & (x_1, x_n)_{n,0} & (x_1, y)_{n,0} \\ \vdots & \ddots & \vdots & \vdots \\ (x_n, x_1)_{n,0} & \cdots & (x_n, x_n)_{n,0} & (x_n, y)_{n,0} \\ (y, x_1)_{n,0} & \cdots & (y, x_n)_{n,0} & (y, y)_{n,0} \end{pmatrix} \in M_{n+1}(E) \ ,$$

where $(,)_{n,0}$ is the Hermitian form on Z_n defined by

$$(x, x')_{n,0} = {}^t x \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \overline{x}'$$

for $x, x' \in Z_n$. Let $\mathbf{P}$ be the natural projection from Z_n onto Y_n. We also use the following notation for elements in $G_n^*(F)$.

$$m(A, \varepsilon) = \begin{pmatrix} A & & \\ & \varepsilon & \\ & & {}^t\overline{A}^{-1} \end{pmatrix} \qquad (A \in GL_n(E),\ \ \varepsilon \in E^\times,\ \mathrm{N}_{E/F}(\varepsilon) = 1),$$

$$n(\alpha, B) = \begin{pmatrix} 1_n & \alpha & -\frac{1}{2}\alpha {}^t\overline{\alpha} \\ & 1 & -{}^t\overline{\alpha} \\ & & 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0 & B \\ & 1 & 0 \\ & & 1_n \end{pmatrix} \qquad \begin{pmatrix} \alpha \in E^n \\ B = -{}^t\overline{B} \in M_n(E) \end{pmatrix}.$$

Then we have the following formula: for $f \in \mathcal{S}((Z_n)^n \oplus Y_n)$,

$$\omega(m(A,\varepsilon))f(\vec{x};y) = \nu(\varepsilon)^n \nu(\det A)^{2n} |\det A|_E^n f\Big(\sum_{i=1}^n \overline{a}_{i1}x_i, \dots, \sum_{i=1}^n \overline{a}_{in}x_i; \overline{\varepsilon}y\Big),$$

$$\omega(n(\alpha,B))f(\vec{x};y) = \mu(\mathrm{Tr}(B_\alpha \mathrm{Gr}^*_{n+1}(\vec{x};y)))\mu(\mathrm{Tr}(B\mathrm{Gr}_n(\vec{x})))f\Big(\vec{x}; \sum_{i=1}^n \overline{\alpha}_i \mathbf{P} x_i + y\Big),$$

where $A = (a_{ij})$ and $\alpha = (\alpha_i)$. If $f \in \mathcal{S}((Z_n)^n \oplus Y_n)$ is of the form $f = f_1 \otimes f_0$, $f_1 \in \mathcal{S}((Z_n)^n)$, $f_0 \in \mathcal{S}(Y_n)$, then we also have the formula

$$\omega(g)f(\vec{x};y) = \nu(\det g)^n f_1(g^{-1}x_1, \dots, g^{-1}x_n)\omega^n_{\mu,\nu}(g)f_0(y) \qquad (g \in G_n(F)).$$

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathcal{O}_E^\times)^n$ and $\beta = (\beta_1, \dots, \beta_{n-1}, \beta_n) \in (\mathcal{O}_E^\times)^{n-1} \times \mathcal{O}^\times$. We define unramified principal characters ψ^*_α and ψ_β of $U^*_n(F)$ and $U_n(F)$, respectively, by

$$\psi^*_\alpha(u^*) = \mu_E(\alpha_1 u^*_{12} + \alpha_2 u^*_{23} + \cdots + \alpha_n u^*_{n\,n+1}),$$
$$\psi_\beta(u) = \mu_E(\beta_1 u_{12} + \beta_2 u_{23} + \cdots + \beta_{n-1} u_{n-1\,n})\mu(-\beta_n u_{n\,2n})$$

for $u^* = (u^*_{ij}) \in U^*_n(F)$ and $u = (u_{ij}) \in U_n(F)$. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathcal{O}_E^\times)^n$, we put

$$\tilde{\alpha} = (\overline{\alpha}_1, \dots, \overline{\alpha}_{n-1}, \mathrm{N}_{E/F}(\alpha_n)) \in (\mathcal{O}_E^\times)^{n-1} \times \mathcal{O}^\times.$$

In the following, we fix a pair $(\psi^*_\alpha, \psi_{\tilde{\alpha}})$ of unramified principal characters.

Let Δ^*_n be a subgroup of U^*_n of the form

$$\Delta^*_n(F) = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^t\overline{A}^{-1} \end{pmatrix} : A = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL_n(E) \right\}.$$

For each $f \in \mathcal{S}((Z_n)^n \oplus Y_n)$, we define the function $\Psi(f)(g)$ in $g \in G_n(F)$ by

$$\Psi(f)(g) = \int_{\Delta^*_n(F)} \psi^*_\alpha(\delta)^{-1} \omega(\delta \dot{g}) f(e_1, \dots, e_n; \alpha_n f_n) d\delta.$$

Let $W \in \mathrm{Ind}_{U_n(F)}^{G_n(F)} \psi_{\tilde{\alpha}}$. Then an unramified factor of the formula in [6, Corollary 5.5] is given by

$$(W, \Psi(\omega(h)f)) = \int_{U_n(F)\backslash G_n(F)} W(g)\Psi(\omega(h)f)(g)dg.$$

Since $\Psi(f)$ has a compact support in $G_n(F)$ modulo $U_n(F)$ (cf. Lemma (2.1)), the integral reduces to finite sum. Furthermore, as a function in $h \in G_n^*$, $(W, \Psi(\omega(h)f))$ is contained in $\mathrm{Ind}_{U_n^*(F)}^{G_n^*(F)}\psi_\alpha^*$. Therefore, we have a correspondence

$$\mathrm{Ind}_{U_n(F)}^{G_n(F)}\psi_{\tilde{\alpha}} \times \mathcal{S}((Z_n)^n \oplus Y_n) \to \mathrm{Ind}_{U_n^*(F)}^{G_n^*(F)}\psi_\alpha^* .$$

Let f_0 be the characteristic function of the standard $\mathcal{O}_E$-lattice $(Z_n(\mathcal{O}_E))^n \oplus Y_n(\mathcal{O}_E)$. Since f_0 is $G_n^*(\mathcal{O}) \times G_n(\mathcal{O})$-invariant, $(W_{\mathbf{z}}, \Psi(\omega(h)f_0))$ is also $G_n^*(\mathcal{O})$-invariant. The purpose of this section is to calculate $(W_{\mathbf{z}}, \Psi(\omega(h)f_0))$ and determine the associated Satake parameter. We start with calculation of $\Psi(\omega(h)f_0)(g)$.

Lemma 2.1: *Let $\mathbf{k} = (k_1, \ldots, k_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be in $\mathbb{Z}^n$. If $p_1 \geq k_1 \geq p_2 \geq k_2 \geq \cdots \geq p_n \geq k_n \geq 0$, then*

$$\Psi(\omega(t_{\mathbf{p}}^*)f_0)(t_{\mathbf{k}}) = \nu(\varpi)^{k_1+\cdots+k_n}\delta_n^*(t_{\mathbf{p}}^*)^{1/2}\delta_n(t_{\mathbf{k}})^{1/2} .$$

Otherwise, $\Psi(\omega(t_{\mathbf{p}}^*)f_0)(t_{\mathbf{k}})$ equals 0.

Proof: Let φ_0 be the characteristic function of $\mathcal{O}_E$. For $\mathbf{k} \in \mathbf{Z}^n$, put $d(\mathbf{k}) = k_1 + \cdots + k_n$. By definition,

$$\Psi(\omega(t_{\mathbf{p}}^*)f_0)(t_{\mathbf{k}}) = \int_{\Delta_n^*(F)} \psi_\alpha^*(u)^{-1}\omega(ut_{\mathbf{p}}^* \cdot t_{\mathbf{k}})f_0(e_1, \cdots, e_n; \alpha_n f_n)du$$

$$= \nu(\varpi)^{2nd(\mathbf{p})+d(\mathbf{k})}|\varpi|_E^{nd(\mathbf{p})+d(\mathbf{k})/2}\int_{\Delta_n^*(F)} \psi_\alpha^*(u)^{-1}f_0(x_1, \cdots, x_n; \alpha_n\varpi^{k_n}f_n)du ,$$

where

$$x_j = \sum_{i=1}^{j-1}\overline{u}_{ij}\varpi^{p_j-k_i}e_i + \varpi^{p_j-k_j}e_j .$$

This integral equals

$$\varphi_0(\alpha_n\varpi^{k_n})\prod_{j=1}^{n}\varphi_0(\varpi^{p_j-k_j})\prod_{j=2}^{n}\prod_{i=1}^{j-2}\int_E \varphi_0(\overline{u}_{ij}\varpi^{p_j-k_i})du_{ij}$$

$$\times \prod_{j=2}^{n}\int_E \mu_E(\alpha_{j-1}u_{j-1\,j})^{-1}\varphi_0(\overline{u}_{j-1\,j}\varpi^{p_j-k_{j-1}})du_{j-1\,j}$$

$$= \left(\prod_{j=2}^{n}\prod_{i=1}^{j-1}|\varpi|_E^{k_i-p_j}\right)\left(\prod_{j=1}^{n}\varphi_0(\varpi^{p_j-k_j})\right)\left(\prod_{j=2}^{n}\varphi_0(\varpi^{k_{j-1}-p_j})\right)\varphi_0(\varpi^{k_n}) .$$

This implies the assertion. ∎

PROPOSITION 2.2: *Let $W_{\mathbf{z}}$ be the unramified Whittaker function for $\mathbf{z} \in (\mathbb{C}^\times)^n$. Then*

$$\left(\prod_{i=1}^{n}(1+q_E^{-1}z_i)\right)(W_{\mathbf{z}}, \Psi(\omega(h)f_0)) = W^*_{-\mathbf{z}}(h) .$$

Proof: Let $\mathbf{p} \in \mathbf{\Lambda_n}$. We remember that $\nu(\varpi) = -1$. By the formula (1.1) and Lemma 2.1, $(W_{\mathbf{z}}, \Psi(\omega(t^*_{\mathbf{p}})f_0))$ equals

$$\frac{\zeta(\mathbf{z})}{\epsilon(\mathbf{z})}\delta^*_n(t^*_{\mathbf{p}})^{1/2}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}\left\{\sum_{p_i\geq k_i\geq p_{i+1}} K(z_{\sigma(i)}, k_i+(n-i)+\kappa)\nu(\varpi)^{k_i}\right\},$$

where we put $p_{n+1} = 0$ for convenience. We use the following simple formula: For given integers $a \geq b \geq 0$,

$$\sum_{a\geq j\geq b}\nu(\varpi)^j K(z, j+m) = \frac{\nu(\varpi)^b K(z, b+m-1/2) + \nu(\varpi)^a K(z, a+m+1/2)}{z^{1/2}+z^{-1/2}},$$

and

$$\epsilon^*(-\mathbf{z}) = \epsilon(\mathbf{z})\nu(\varpi)^{n(n+1)/2}\prod_{i=1}^{n}(z_i^{1/2}+z_i^{-1/2}), \qquad \zeta^*(-\mathbf{z}) = \zeta(\mathbf{z})\prod_{i=1}^{n}(1+q_E^{-1}z_i) .$$

Therefore, $\prod_{i=1}^{n}(1+q_E^{-1}z_i)(W_{\mathbf{z}}, \Psi(\omega(t^*_{\mathbf{p}})f_0))$ equals

$$\frac{\zeta^*(-\mathbf{z})}{\epsilon^*(-\mathbf{z})}\delta^*(t^*_{\mathbf{p}})^{1/2}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)$$
$$\times\prod_{i=1}^{n}\left(K(-z_{\sigma(i)}, p_i+(n-i)+1) - K(-z_{\sigma(i)}, p_{i+1}+(n-i))\right) .$$

Since the sum over S_n equals the determinant of the matrix

$$\begin{pmatrix} K_{11}-K_{21} & K_{12}-K_{22} & \cdots & K_{1n}-K_{2n} \\ K_{21}-K_{31} & K_{22}-K_{32} & \cdots & K_{2n}-K_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1\,1}-K_{n1} & K_{n-1\,2}-K_{n2} & \cdots & K_{n-1\,n}-K_{nn} \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{pmatrix},$$

where $K_{ij} = K(-z_j, p_i+(n-i)+1)$, it is also equal to

$$\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}K(-z_{\sigma(i)}, p_i+(n-i)+1) .$$

This implies the assertion. ∎

Let $\mathcal{H}_n^{(*)}$ be the convolution algebra consisting of all locally constant and compactly supported functions on $G_n^{(*)}(F)$. The characteristic function $\xi_n^{(*)}$ of $G_n^{(*)}(\mathcal{O})$ is an idempotent element in $\mathcal{H}_n^{(*)}$ and $\omega(\xi_n^{(*)})$ defines a projection from $\mathcal{S}((Z_n)^n \oplus Y_n)$ to the subspace $\mathcal{S}((Z_n)^n \oplus Y_n)^{\omega(G_n^{(*)}(\mathcal{O}))}$ of $\omega(G_n^{(*)}(\mathcal{O}))$-invariant elements. By [3, Theorem 10.2] (or [5, Chapitre 5, Théorème I.4]), it is known that the subspace $\mathcal{S}((Z_n)^n \oplus Y_n)^{\omega(G_n(\mathcal{O}))}$ coincides with the subspace $\omega(\mathcal{H}_n^*)f_0$. Therefore, for each $f \in \mathcal{S}((Z_n)^n \oplus Y_n)$, there exists $\varphi_f \in \mathcal{H}_n^*$ such that $\omega(\xi_n)f = \omega(\varphi_f)f_0$. Then we have

$$c(\mathbf{z})(W_{\mathbf{z}}, \Psi(f)) = c(\mathbf{z})(W_{\mathbf{z}}, \Psi(\omega(\xi_n)f)) = c(\mathbf{z})(W_{\mathbf{z}}, \Psi(\omega(\varphi_f)f_0)) = \varphi_f * W^*_{-\mathbf{z}} \ ,$$

where $c(\mathbf{z}) = \prod_{i=1}^n (1 + q_E^{-1} z_i)$. Hence we obtain a map

$$A_{\mathbf{z}}\colon \mathbf{W}_{\mathbf{z}}(\psi_{\tilde{\alpha}}) \times \mathcal{S}((Z_n)^n \oplus Y_n) \to \mathbf{W}^*_{-\mathbf{z}}(\psi_\alpha)\colon (W, f) \mapsto c(\mathbf{z})(W, \Psi(\omega(\cdot)f)) \ .$$

If $\mathbf{z} \in \Omega_n$, then $A_{\mathbf{z}}$ is non-zero.

THEOREM 2.3: *For any irreducible spherical representation $\pi_{\mathbf{z}}$, one has*

$$\mathrm{Hom}_{G_n^*(F) \times G_n(F)}(\omega, \pi^*_{-\mathbf{z}} \otimes \pi_{\mathbf{z}}) \neq 0 \ .$$

In other words, $\pi_{\mathbf{z}} \mapsto \pi^*_{-\mathbf{z}}$ is the local Howe correspondence with respect to $\omega = \omega_{\mu,\nu}^{n(2n+1)}$.

Proof: It is sufficient to consider $\pi_{\mathbf{z}}$ for $\mathbf{z} \in \Omega_n$. As we noted in Section 1, $\pi_{\mathbf{z}}$ (resp. $\pi^*_{-\mathbf{z}}$) is isomorphic to the unique irreducible quotient of $\mathbf{W}_{\mathbf{z}}(\psi_{\tilde{\alpha}})$ (resp. $\mathbf{W}^*_{-\mathbf{z}}(\psi_\alpha)$). We denote by $\mathbf{V}_{\mathbf{z}}$ the kernel of the quotient map form $\mathbf{W}_{\mathbf{z}}(\psi_{\tilde{\alpha}})$ to $\pi_{\mathbf{z}}$. Let $\widetilde{A}_{\mathbf{z}}$ be the composition of $A_{\mathbf{z}}$ and the quotient map from $\mathbf{W}^*_{-\mathbf{z}}(\psi_\alpha)$ to $\pi^*_{-\mathbf{z}}$. Since $A_{\mathbf{z}}$ is surjective, so is $\widetilde{A}_{\mathbf{z}}$. We set

$$\mathbf{V'}_{\mathbf{z}} = \{W \in \mathbf{W}_{\mathbf{z}}(\psi_{\tilde{\alpha}})\colon \widetilde{A}_{\mathbf{z}}(W, f) = 0 \ \text{ for all } \ f \in \mathcal{S}((Z_n)^n \oplus Y_n)\} \ .$$

Since $\mathbf{V'}_{\mathbf{z}}$ is a proper $G_n(F)$-invariant subspace, we have $\mathbf{V'}_{\mathbf{z}} \subset \mathbf{V}_{\mathbf{z}}$. We suppose $\mathbf{V'}_{\mathbf{z}} \neq \mathbf{V}_{\mathbf{z}}$. Then there exists a non-zero irreducible subspace $\mathbf{U}$ of $\mathbf{V}_{\mathbf{z}}/\mathbf{V}'_{\mathbf{z}}$, and the restriction of $\widetilde{A}_{\mathbf{z}}$ to $\mathbf{U}$ gives rise to a non-zero $G_n^*(F) \times G_n(F)$-morphism from $\mathcal{S}((Z_n)^n \otimes Y_n)$ onto $\pi^*_{-\mathbf{z}} \otimes \mathbf{U}^\vee$, where $\mathbf{U}^\vee$ denotes the smooth dual of $\mathbf{U}$. Thus $\pi^*_{-\mathbf{z}} \to \mathbf{U}^\vee$ is the local Howe correspondence. However, $\mathbf{U}^\vee$ is not spherical since the space $\mathbf{V_z}/\mathbf{V'_z}$ never has a $G_n(\mathcal{O})$-invariant vector. This contradicts a result of Howe [3, Theorem 7.1 (b)]. Therefore, we have $\mathbf{V_z} = \mathbf{V'_z}$. Then $\widetilde{A}_{\mathbf{z}}$ induces a map from $\pi_{\mathbf{z}} \times \mathcal{S}((Z_n)^n \oplus Y_n)$ onto $\pi^*_{-\mathbf{z}}$, and hence we have a non-zero $G^*_n(F) \times G_n(F)$-morphism from $\mathcal{S}((Z_n)^n \otimes Y_n)$ onto $\pi^*_{-\mathbf{z}} \otimes \pi^\vee_{\mathbf{z}}$, where $\pi^\vee_{\mathbf{z}}$ denotes the contragradient representation of $\pi_{\mathbf{z}}$. Then the equivalence $\pi^\vee_{\mathbf{z}} \cong \pi_{\mathbf{z}^{-1}} \cong \pi_{\mathbf{z}}$ implies the assertion. ∎

We note that $\pi^*_{-\mathbf{z}}$ is not necessarily generic even if $\pi_{\mathbf{z}}$ is generic. Such a case occurs if and only if $\mathbf{z} \in \Omega_n$ satisfies $\zeta(\mathbf{z}^{-1}) \neq 0$ and $c(\mathbf{z}^{-1}) = 0$. For example, if $n = 1$ and $\mathbf{z} = -q_E^{-1} \in \Omega_1$, then $\pi_{\mathbf{z}} = I_{\mathbf{z}}$ is generic, but $\pi^*_{-\mathbf{z}}$ is the trivial representation.

3. The local theta correspondence from G^*_n to G_{n+1}

In this section, we consider the space $Z^*_n \otimes Z_{n+1}$ equipped with skew Hermitian form $(,)_n \otimes \langle,\rangle_{n+1}$. In a similar fashion as Section 2, the Weil representation $\omega^{(n+1)(2n+1)}_{\mu,\nu}$ is restricted to $G^*_n(F) \otimes G_{n+1}(F)$. We also write simply ω for $\omega^{(n+1)(2n+1)}_{\mu,\nu}$. Let $Y_{(n+1)(2n+1)}$ be a totally isotropic subspace of the form

$$Y_{(n+1)(2n+1)} = Z^*_n \otimes Y_{n+1} = \bigoplus_{i=1}^{n+1} Z^*_n \otimes f_i \ ,$$

which is identified with $(Z^*_n)^{n+1}$. The action of $G^*_n(F) \times G_{n+1}(F)$ on $\mathcal{S}((Z^*_n)^{n+1})$ is given as follows. For $f \in \mathcal{S}((Z^*_n)^{n+1})$ and $\vec{x} = (x_1, \ldots, x_{n+1}) \in (Z^*_n)^{n+1}$,

$$\omega(h) f(\vec{x}) = \nu(\det h)^{n+1} f(h^{-1}x_1, \ldots, h^{-1}x_{n+1}) \ ,$$

$$\omega\left(\begin{pmatrix} A & 0 \\ 0 & {}^t\overline{A}^{-1} \end{pmatrix}\right) f(\vec{x}) = \nu(\det A)^{2n+1} |\det A|_E^{n+1/2} \times f\Big(\big(\sum_{i=1}^{n+1} \overline{a}_{i1} x_i, \ldots, \sum_{i=1}^{n+1} \overline{a}_{i\,n+1} x_i\big)\Big) \ ,$$

$$\omega\left(\begin{pmatrix} 1_{n+1} & B \\ 0 & 1_{n+1} \end{pmatrix}\right) f(\vec{x}) = \mu(\mathrm{Tr}(B \cdot \mathrm{Gr}^+_{n+1}(\vec{x}))) f(\vec{x}) \ ,$$

where $h \in G^*_n(F)$, $A = (a_{ij}) \in GL_{n+1}(E)$ and $B = {}^t\overline{B} \in M_{n+1}(E)$, and we put

$$\mathrm{Gr}^+_{n+1}(\vec{x}) = \begin{pmatrix} (x_1, x_1)_n & \cdots & (x_1, x_{n+1})_n \\ \vdots & \ddots & \vdots \\ (x_{n+1}, x_1)_n & \cdots & (x_{n+1}, x_{n+1})_n \end{pmatrix} .$$

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathcal{O}_E^\times)^n$, the unramified principal character $\psi_{\hat{\alpha}}$ of $U_{n+1}(F)$ is defined to be

$$\psi_{\hat{\alpha}}(u) = \mu_E(\overline{\alpha}_1 u_{12} + \cdots + \overline{\alpha}_{n-1} u_{n-1\,n} - u_{n\,n+1})\mu(\mathrm{N}_{E/F}(\alpha_n) u_{n+1\,2(n+1)})$$

for $u = (u_{ij}) \in U_{n+1}(F)$. Throughout this section, we fix a pair $(\psi_\alpha^*, \psi_{\hat{\alpha}})$ of unramified principal characters.

Let Δ_{n+1} be a subgroup of U_{n+1} of the form

$$\Delta_{n+1}(F) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t\overline{A}^{-1} \end{pmatrix} : A = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(E) \right\} .$$

For each $f \in \mathcal{S}((Z_n^*)^{n+1})$, we define the function $\Psi^*(f)(h)$ in $h \in G_n^*(F)$ by

$$\Psi^*(f)(h) = \int_{\Delta_{n+1}(F)} \psi_{\hat{\alpha}}(\delta)^{-1} \omega(h \cdot \delta) f(e_1^*, \ldots, e_n^*, \alpha_n e_0^*) d\delta .$$

Let $W^* \in \mathrm{Ind}_{U_n^*(F)}^{G_n^*(F)} \psi_\alpha$. Then an unramified factor of the formula in [6, Corollary 4.5] is given by

$$(W^*, \Psi^*(\omega(g)f)) = \int_{U_n^*(F)\backslash G_n^*(F)} W^*(h) \Psi^*(\omega(g)f)(h) dh .$$

Since $(W^*, \Psi^*(\omega(\cdot)f))$ is contained in $\mathrm{Ind}_{U_{n+1}(F)}^{G_{n+1}(F)} \psi_{\hat{\alpha}}$, we obtain a correspondence

$$\mathrm{Ind}_{U_n^*(F)}^{G_n^*(F)} \psi_\alpha^* \times \mathcal{S}((Z_n^*)^{n+1}) \to \mathrm{Ind}_{U_{n+1}(F)}^{G_{n+1}(F)} \psi_{\hat{\alpha}} .$$

Let f_0^* be the characteristic function of the standard $\mathcal{O}_E$-lattice $(Z_n^*(\mathcal{O}_E))^{n+1}$. In like manner as Section 2, we have the following:

LEMMA 3.1: *Let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $\mathbf{p} = (p_1, \ldots, p_{n+1}) \in \mathbb{Z}^{n+1}$. If $p_1 \geq k_1 \geq p_2 \geq k_2 \geq \cdots \geq p_n \geq k_n \geq p_{n+1} \geq 0$, then*

$$\Psi^*(\omega(t_{\mathbf{p}}) f_0^*)(t_{\mathbf{k}}^*) = \nu(\varpi)^{p_1 + p_2 + \cdots + p_{n+1}} \delta_{n+1}(t_{\mathbf{p}})^{1/2} \delta_n^*(t_{\mathbf{k}}^*)^{1/2} .$$

Otherwise, $\Psi^*(\omega(t_{\mathbf{p}}) f_0^*)(t_{\mathbf{k}}^*)$ equals 0.

PROPOSITION 3.2: *Let $W^*_{\mathbf{z}}$ be the unramified Whittaker function for $\mathbf{z} \in (\mathbf{C}^\times)^n$. Let $-(\mathbf{z},1) = (-z_1,\dots,-z_n,-1) \in (\mathbf{C}^\times)^{n+1}$. Then*

$$(-1)^n(1+q^{-1})\left(\prod_{i=1}^{n}(1-q_E^{-1}z_i)\right)(W^*_{\mathbf{z}},\Psi^*(\omega(g)f_0^*)) = W_{-(\mathbf{z},1)}(g).$$

Proof: Let $\mathbf{p} \in \Lambda_{n+1}$. It follows from Lemma 3.1 that $(W^*_{\mathbf{z}},\Psi^*(\omega(t_{\mathbf{p}})f_0^*))$ equals

$$\frac{\zeta^*(\mathbf{z})}{\epsilon^*(\mathbf{z})}\delta_{n+1}(t_{\mathbf{p}})^{1/2}\nu(\varpi)^{d(\mathbf{p})}$$
$$\times \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}\left\{\sum_{p_i\ge k_i\ge p_{i+1}} K(z_{\sigma(i)},k_i+(n-i)+1)\right\}.$$

For $z \in \mathbb{C}^\times$ and $k \in \mathbb{Z}$, we put $K^+(z,k/2) = z^{k/2}+z^{-k/2}$. Then we have

$$\sum_{a\ge j\ge b} K(z,j+m) = \frac{K^+(z,b+m-1/2)-K^+(z,a+m+1/2)}{K(z,1/2)}$$

for integers $a \ge b \ge 0$. Therefore, the sum over S_n equals

$$\prod_{i=1}^{n}\frac{1}{K(z_i,1/2)}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)$$
$$\times\prod_{i=1}^{n}\left(K^+(z_{\sigma(i)},p_{i+1}+(n-i)+1/2)-K^+(z_{\sigma(i)},p_i+(n-i)+3/2)\right).$$

By calculation of determinants, we obtain

$$\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}\left(K^+\big(z_{\sigma(i)},p_{i+1}+(n-i)+\frac{1}{2}\big)-K^+\big(z_{\sigma(i)},p_i+(n-i)+\frac{3}{2}\big)\right)$$
$$=\frac{\nu(\varpi)^{n^2/2+d(\mathbf{p})}}{K(-1,1/2)}\sum_{\sigma\in S_{n+1}}\operatorname{sgn}(\sigma)\prod_{i=1}^{n+1}K\big(z'_{\sigma(i)},p_i+(n+1-i)+\frac{1}{2}\big),$$

where $(z'_1,\dots,z'_{n+1}) = -(\mathbf{z},1)$. As a consequence, $(W^*_{\mathbf{z}},\Psi^*(\omega(t_{\mathbf{p}})f_0^*))$ equals

$$\frac{\zeta^*(\mathbf{z})}{\epsilon^*(\mathbf{z})}\delta_{n+1}(t_{\mathbf{p}})^{1/2}\frac{\nu(\varpi)^{n^2/2}}{K(-1,1/2)}\prod_{i=1}^{n}\frac{1}{K(z_i,1/2)}$$
$$\times\sum_{\sigma\in S_{n+1}}\operatorname{sgn}(\sigma)\prod_{i=1}^{n+1}K(z'_{\sigma(i)},p_i+(n+1-i)+1/2).$$

Furthermore, we have

$$\epsilon(-(\mathbf{z},1)) = \epsilon^*(\mathbf{z})\nu(\varpi)^{n(n+2)/2}K(-1,1/2)\prod_{i=1}^{n}K(z_i,1/2)\,,$$

$$\zeta(-(\mathbf{z},1)) = \zeta^*(\mathbf{z})(1+q^{-1})\prod_{i=1}^{n}(1-q_E^{-1}z_i)\,.$$

This completes the proof. ∎

The same argument as in Section 2 gives the following:

THEOREM 3.3: *For any irreducible spherical representation $\pi_{\mathbf{z}}^*$, one has*

$$\mathrm{Hom}_{G_n^*(F)\times G_{n+1}(F)}(\omega,\pi_{\mathbf{z}}^*\otimes\pi_{-(\mathbf{z},1)}) \neq 0\,.$$

In other words, $\pi_{\mathbf{z}}^* \mapsto \pi_{-(\mathbf{z},1)}$ is the local Howe correspondence with respect to $\omega = \omega_{\mu,\nu}^{(n+1)(2n+1)}$.

Since $\zeta^*(\mathbf{z})\zeta^*(\mathbf{z}^{-1}) \neq 0$ implies $\zeta(-(\mathbf{z},1))\zeta(-(\mathbf{z},1)^{-1}) \neq 0$, $\pi_{-(\mathbf{z},1)}$ is also generic if $\pi_{\mathbf{z}}^*$ is generic.

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